

# Module 5 Cables and Arches

Version 2 CE IIT, Kharagpur

# Lesson

# 32

# Three Hinged Arch

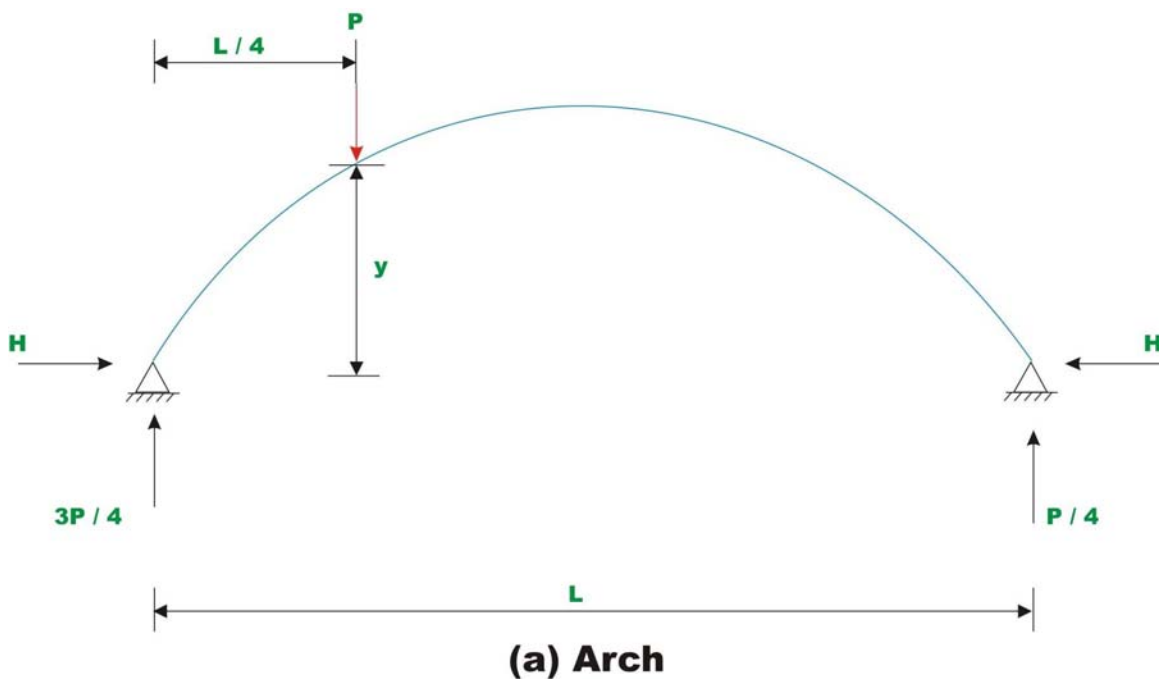
## Instructional Objectives:

After reading this chapter the student will be able to

1. Define an arch.
2. Identify three-hinged, two-hinged and hingeless arches.
3. State advantages of arch construction.
4. Analyse three-hinged arch.
5. Evaluate horizontal thrust in three-hinged arch.

## 32.1 Introduction

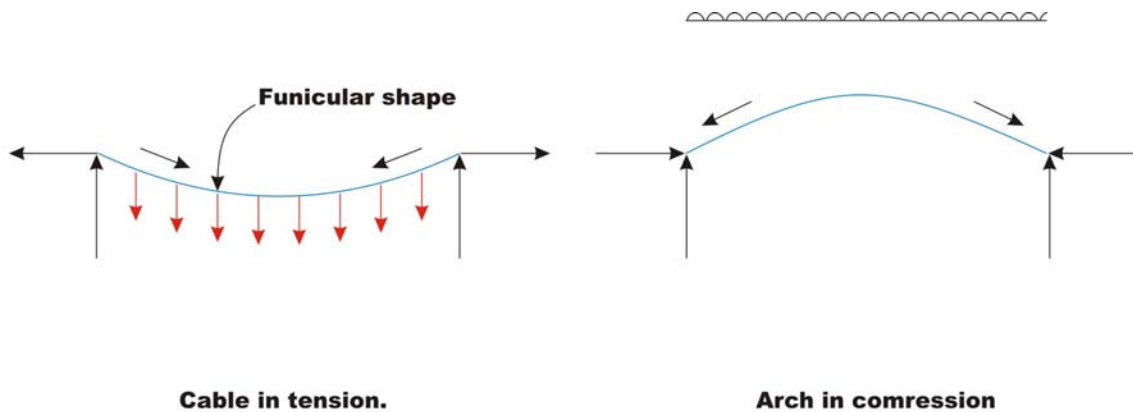
In case of beams supporting uniformly distributed load, the maximum bending moment increases with the square of the span and hence they become uneconomical for long span structures. In such situations arches could be advantageously employed, as they would develop horizontal reactions, which in turn reduce the design bending moment.



**Fig. 32.1 Beam and Arch comparison.**

For example, in the case of a simply supported beam shown in Fig. 32.1, the bending moment below the load is  $\frac{3PL}{16}$ . Now consider a two hinged symmetrical arch of the same span and subjected to similar loading as that of simply supported beam. The vertical reaction could be calculated by equations of statics. The horizontal reaction is determined by the method of least work. Now

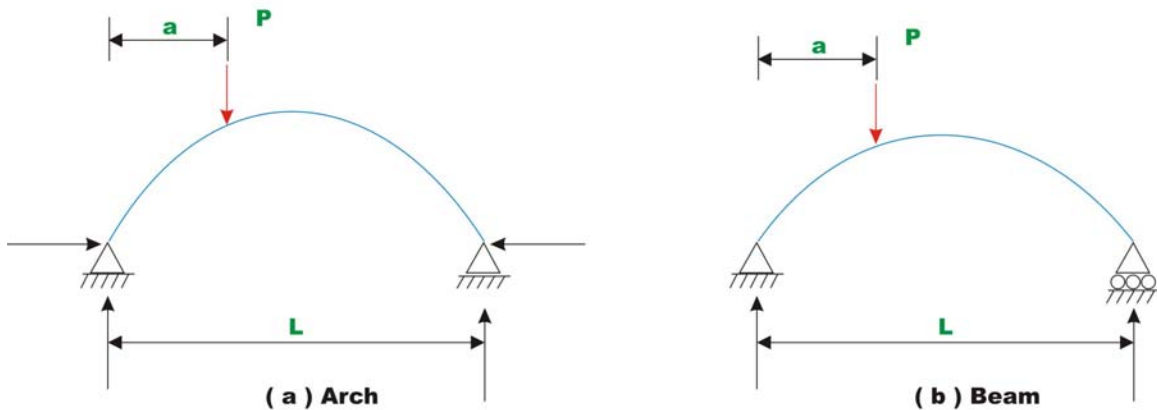
the bending moment below the load is  $\frac{3PL}{16} - Hy$ . It is clear that the bending moment below the load is reduced in the case of an arch as compared to a simply supported beam. It is observed in the last lesson that, the cable takes the shape of the loading and this shape is termed as funicular shape. If an arch were constructed in an inverted funicular shape then it would be subjected to only compression for those loadings for which its shape is inverted funicular.



**Fig. 32.2 Cable and Arch structure.**

Since in practice, the actual shape of the arch differs from the inverted funicular shape or the loading differs from the one for which the arch is an inverted funicular, arches are also subjected to bending moment in addition to compression. As arches are subjected to compression, it must be designed to resist buckling.

Until the beginning of the 20<sup>th</sup> century, arches and vaults were commonly used to span between walls, piers or other supports. Now, arches are mainly used in bridge construction and doorways. In earlier days arches were constructed using stones and bricks. In modern times they are being constructed of reinforced concrete and steel.

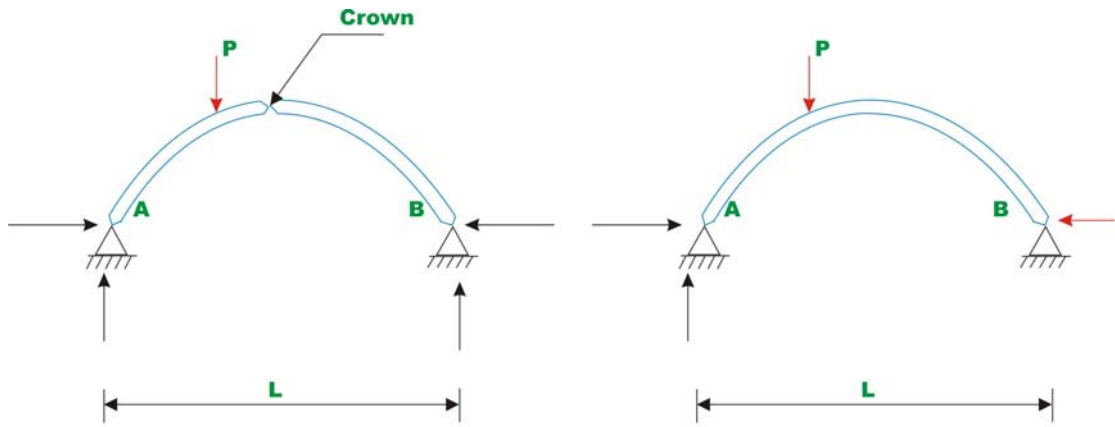


**Fig. 32.3**

A structure is classified as an arch not based on its shape but the way it supports the lateral load. Arches support load primarily in compression. For example in Fig 32.3b, no horizontal reaction is developed. Consequently bending moment is not reduced. It is important to appreciate the point that the definition of an arch is a structural one, not geometrical.

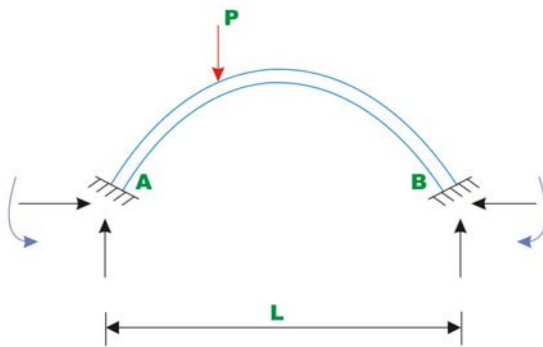
## 32.2 Type of arches

There are mainly three types of arches that are commonly used in practice: three hinged arch, two-hinged arch and fixed-fixed arch. Three-hinged arch is statically determinate structure and its reactions / internal forces are evaluated by static equations of equilibrium. Two-hinged arch and fixed-fixed arch are statically indeterminate structures. The indeterminate reactions are determined by the method of least work or by the flexibility matrix method. In this lesson three-hinged arch is discussed.



**(a) Three - hinged arch**

**(b) Two - hinged arch**

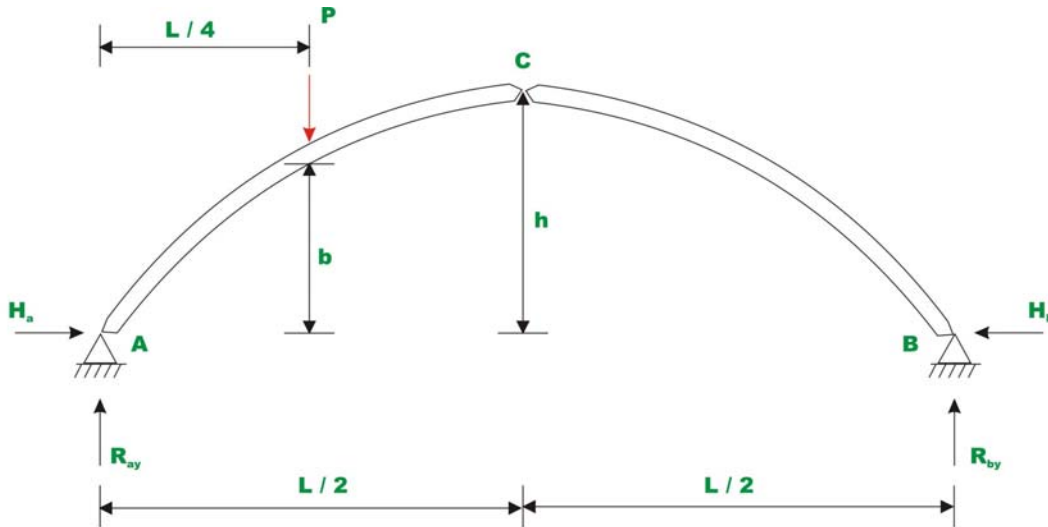


**(c) Fixed hinged arch**

**Fig. 32.4 Types of arches.**

### 32.3 Analysis of three-hinged arch

In the case of three-hinged arch, we have three hinges: two at the support and one at the crown thus making it statically determinate structure. Consider a three hinged arch subjected to a concentrated force  $P$  as shown in Fig 32.5.



**Fig. 32.5 Three hinged arch.**

There are four reaction components in the three-hinged arch. One more equation is required in addition to three equations of static equilibrium for evaluating the four reaction components. Taking moment about the hinge of all the forces acting on either side of the hinge can set up the required equation. Taking moment of all the forces about hinge A, yields

$$R_{by} = \frac{PL}{4L} = \frac{P}{4} \quad (32.1)$$

$$\sum Fy = 0 \quad \Rightarrow \quad R_{ay} = \frac{3P}{4} \quad (32.2)$$

Taking moment of all forces right of hinge C about hinge C leads to

$$\begin{aligned} H_b \times h &= \frac{R_{by}L}{2} \\ \Rightarrow H_b &= \frac{R_{by}L}{2h} = \frac{PL}{8h} \end{aligned} \quad (32.3)$$

Applying  $\sum F_x = 0$  to the whole structure gives  $H_a = \frac{PL}{8h}$

Now moment below the load is given by ,

$$M_D = \frac{R_{ay}L}{4} - H_a b$$
$$M_D = \frac{3PL}{16} - \frac{PLb}{8h} \quad (32.4)$$

$$\text{If } \frac{b}{h} = \frac{1}{2} \quad \text{then} \quad M_D = \frac{3PL}{16} - \frac{PL}{16} = 0.125PL \quad (32.5)$$

For a simply supported beam of the same span and loading, moment under the loading is given by,

$$M_D = \frac{3PL}{16} = 0.375PL \quad (32.6)$$

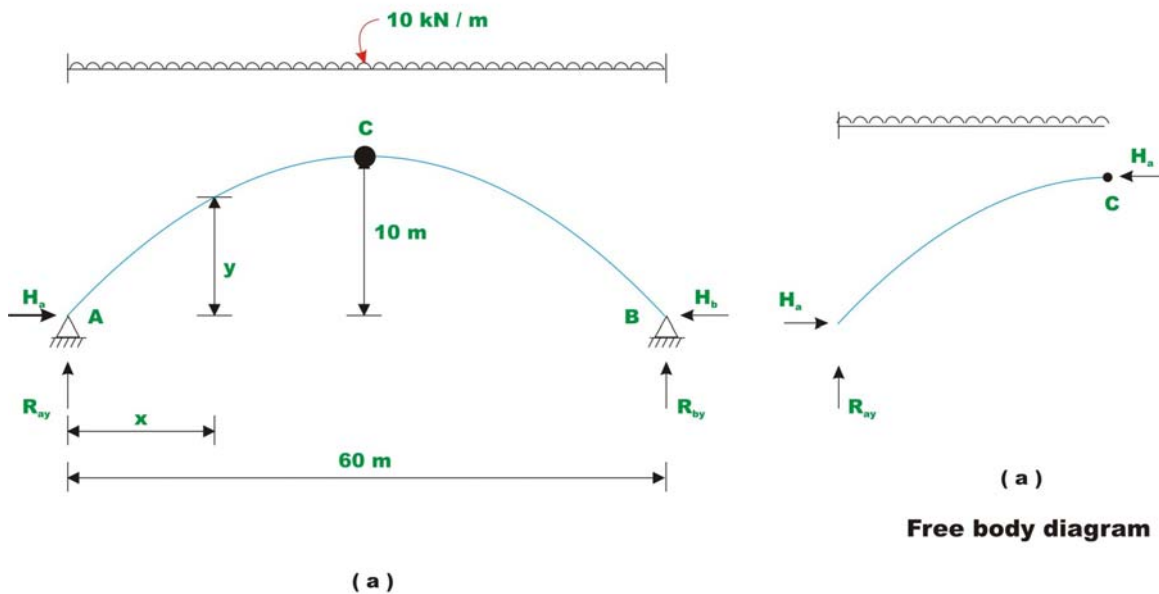
For the particular case considered here, the arch construction has reduced the moment by 66.66 %.

### Example 32.1

A three-hinged parabolic arch of uniform cross section has a span of 60 m and a rise of 10 m. It is subjected to uniformly distributed load of intensity 10 kN/m as shown in Fig. 32.6 Show that the bending moment is zero at any cross section of the arch.



**Solution:**



**Fig. 32.6 Three hinged arch of Example 32.1**

**Reactions:**

Taking moment of all the forces about hinge  $A$ , yields

$$R_{ay} = R_{by} = \frac{10 \times 60}{2} = 300 \text{ kN} \quad (1)$$

Taking moment of forces left of hinge  $C$  about  $C$ , one gets

$$R_{ay} \times 30 - H_a \times 10 - 10 \times 30 \times \frac{30}{2} = 0$$

$$H_a = \frac{300 \times 30 - 10 \times 30 \times \left(\frac{30}{2}\right)}{10} \quad (2)$$

$$= 450 \text{ kN}$$

From  $\sum F_x = 0$  one could write,  $H_b = 450 \text{ kN}$ .

The shear force at the mid span is zero.

**Bending moment**

The bending moment at any section  $x$  from the left end is,

$$M_x = R_{ay}x - H_a y - 10 \frac{x^2}{2} \quad (3)$$

The equation of the three-hinged parabolic arch is

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2 \quad (4)$$

$$M_x = 300x - \left( \frac{2}{3}x - \frac{10}{30^2}x^2 \right) 450 - 5x^2$$

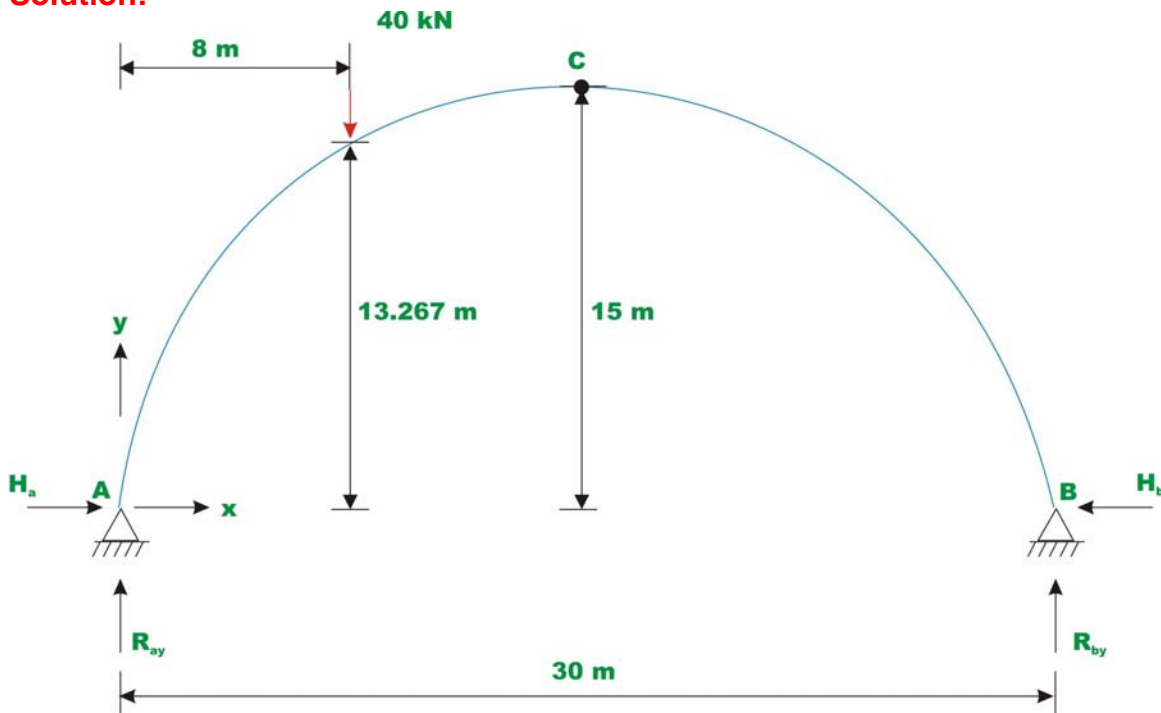
$$= 300x - 300x + 5x^2 - 5x^2 = 0$$

In other words a three hinged parabolic arch subjected to uniformly distributed load is not subjected to bending moment at any cross section. It supports the load in pure compression. Can you explain why the moment is zero at all points in a three-hinged parabolic arch?

### Example 32.2

A three-hinged semicircular arch of uniform cross section is loaded as shown in Fig 32.7. Calculate the location and magnitude of maximum bending moment in the arch.

**Solution:**



**Fig. 32.7 A semi circular arch of Example 32.2**

### Reactions:

Taking moment of all the forces about hinge  $B$  leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.33 \text{ kN } (\uparrow)$$
$$\sum F_y = 0 \quad \Rightarrow R_{by} = 10.67 \text{ kN } (\uparrow) \quad (1)$$

### Bending moment

Now making use of the condition that the moment at hinge  $C$  of all the forces left of hinge  $C$  is zero gives,

$$M_c = R_{ay} \times 15 - H_a \times 15 - 40 \times 7 = 0 \quad (2)$$

$$H_a = \frac{29.33 \times 15 - 40 \times 7}{15} = 10.66 \text{ kN } (\rightarrow)$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 10.66 \text{ kN } (\leftarrow)$$

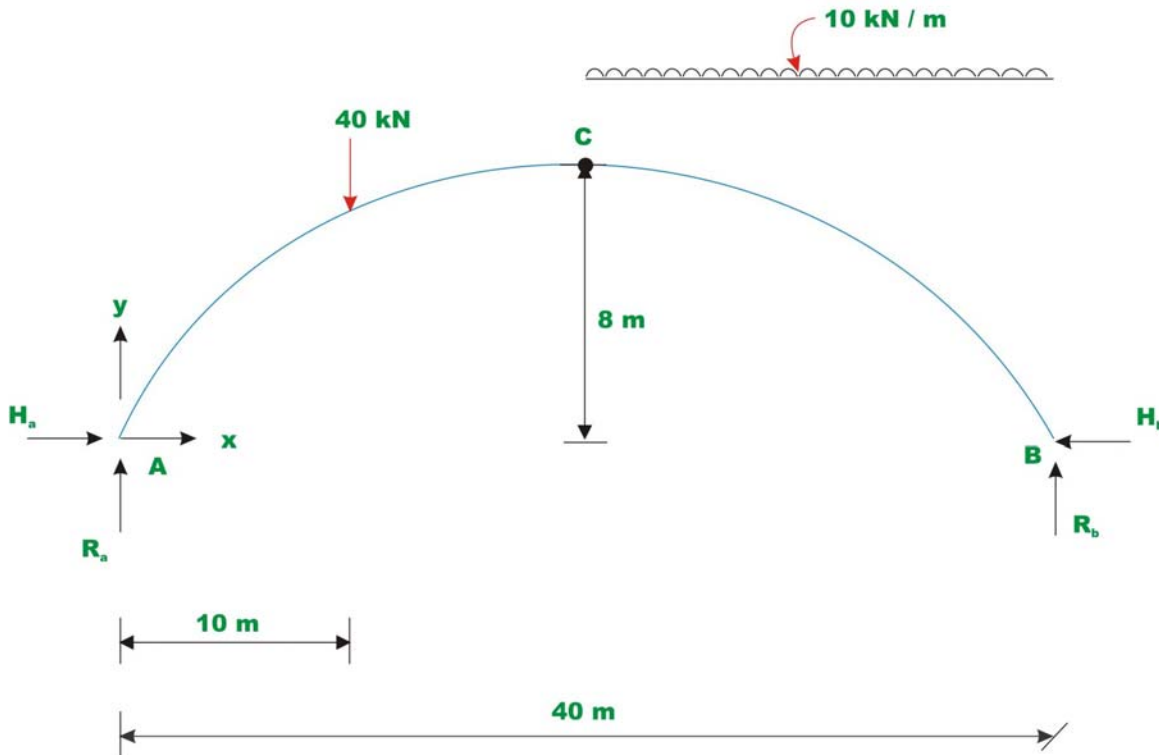
The maximum positive bending moment occurs below  $D$  and it can be calculated by taking moment of all forces left of  $D$  about  $D$ .

$$M_D = R_{ay} \times 8 - H_a \times 13.267 \quad (3)$$
$$= 29.33 \times 8 - 10.66 \times 13.267 = 93.213 \text{ kN}$$

### Example 32.3

A three-hinged parabolic arch is loaded as shown in Fig 32.8a. Calculate the location and magnitude of maximum bending moment in the arch. Draw bending moment diagram.

**Solution:**



**Fig. 32.8a Example 32.3**

**Reactions:**

Taking  $A$  as the origin, the equation of the three-hinged parabolic arch is given by,

$$y = \frac{8}{10}x - \frac{8}{400}x^2 \quad (1)$$

Taking moment of all the forces about hinge  $B$  leads to,

$$R_{ay} = \frac{40 \times 30 + 10 \times 20 \times \left(\frac{20}{2}\right)}{40} = 80 \text{ kN } (\uparrow)$$

$$\sum F_y = 0 \quad \Rightarrow R_{by} = 160 \text{ kN } (\uparrow) \quad (2)$$

Now making use of the condition that, the moment at hinge  $C$  of all the forces left of hinge  $C$  is zero gives,

$$M_c = R_{ay} \times 20 - H_a \times 8 - 40 \times 10 = 0$$

$$H_a = \frac{80 \times 20 - 40 \times 10}{8} = 150 \text{ kN } (\rightarrow) \quad (3)$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 150 \text{ kN } (\leftarrow) \quad (4)$$

### Location of maximum bending moment

Consider a section  $x$  from end  $B$ . Moment at section  $x$  in part  $CB$  of the arch is given by (please note that  $B$  has been taken as the origin for this calculation),

$$M_x = 160x - \left( \frac{8}{10}x - \frac{8}{400}x^2 \right) 150 - \frac{10}{2}x^2 \quad (5)$$

According to calculus, the necessary condition for extremum (maximum or minimum) is that  $\frac{\partial M_x}{\partial x} = 0$ .

$$\frac{\partial M_x}{\partial x} = 160 - \left( \frac{8}{10} - \frac{8 \times 2}{400}x \right) 150 - 10x$$

$$= 40 - 4x = 0 \quad (6)$$

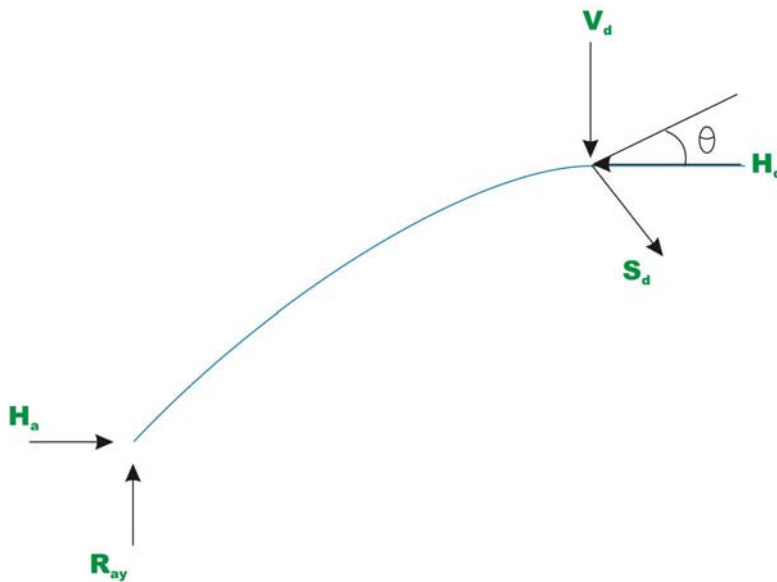
$$x = 10 \text{ m.}$$

Substituting the value of  $x$  in equation (5), the maximum bending moment is obtained. Thus,

$$M_{\max} = 160(10) - \left( \frac{8}{10}(10) - \frac{8}{400}(10)^2 \right) 150 - \frac{10}{2}(10)^2$$

$$M_{\max} = 200 \text{ kN.m.} \quad (7)$$

**Shear force at  $D$  just left of 40 kN load**



**Fig. 32.8b**

The slope of the arch at  $D$  is evaluated by,

$$\tan \theta = \frac{dy}{dx} = \frac{8}{10} - \frac{16}{400}x \quad (8)$$

Substituting  $x = 10$  m. in the above equation,  $\theta_D = 21.8^\circ$

**Shear force  $S_d$  at left of  $D$  is**

$$\begin{aligned} S_d &= H_a \sin \theta - R_{ay} \cos \theta \\ S_d &= 150 \sin(21.80) - 80 \cos(21.80) \\ &= -18.57 \text{ kN.} \end{aligned} \quad (9)$$

### Example 32.4

A three-hinged parabolic arch of constant cross section is subjected to a uniformly distributed load over a part of its span and a concentrated load of 50 kN, as shown in Fig. 32.9. The dimensions of the arch are shown in the figure. Evaluate the horizontal thrust and the maximum bending moment in the arch.

### Solution:

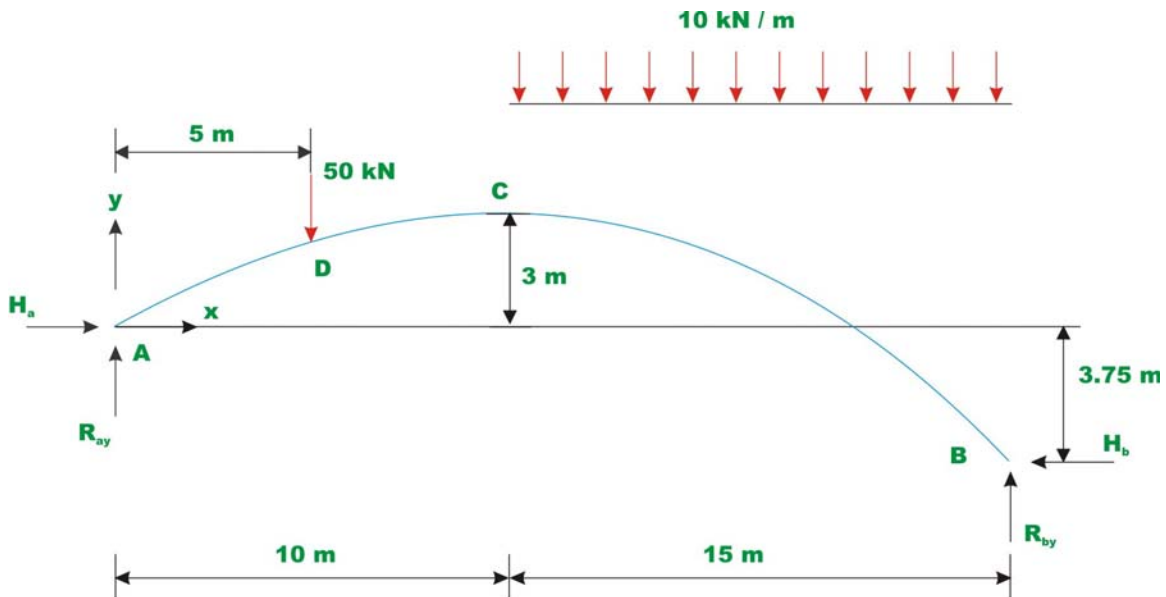


Fig. 32.9

### Reactions:

Taking  $A$  as the origin, the equation of the parabolic arch may be written as,

$$y = -0.03x^2 + 0.6x \quad (1)$$

Taking moment of all the loads about  $B$  leads to,

$$\begin{aligned} R_{ay} &= \frac{1}{25} \left[ 50 \times 20 + 10 \times 15 \times \frac{15}{2} - H_a \times 3.75 \right] \\ &= \frac{1}{25} [2125 - 3.75 H_a] \end{aligned} \quad (2)$$

Taking moment of all the forces right of hinge  $C$  about the hinge  $C$  and setting  $M_c = 0$  leads to,

$$R_{by} \times 15 - 6.75H_b - 10 \times 15 \times \frac{15}{2} = 0$$

$$R_{by} = \frac{1}{15} [1125 + 6.75H_b] \quad (3)$$

Since there are no horizontal loads acting on the arch,

$$H_a = H_b = H \quad (\text{say})$$

Applying  $\sum F_y = 0$  for the whole arch,

$$R_{ay} + R_{by} = 10 \times 15 + 50 = 200$$

$$\frac{1}{25} [2125 - 3.75H] + \frac{1}{15} [1125 + 6.75H] = 200$$

$$85 - 0.15H + 75 + 0.45H = 200$$

$$H = \frac{40}{0.3} = 133.33 \text{ kN} \quad (4)$$

From equation (2),

$$R_{ay} = 65.0 \text{ kN}$$

$$R_{by} = 135.0 \text{ kN} \quad (5)$$

### Bending moment

From inspection, the maximum negative bending moment occurs in the region  $AD$  and the maximum positive bending moment occurs in the region  $CB$ .

### Span AD

Bending moment at any cross section in the span AD is

$$M = R_{ay}x - H_a(-0.03x^2 + 0.6x) \quad 0 \leq x \leq 5 \quad (6)$$

For, the maximum negative bending moment in this region,

$$\frac{\partial M}{\partial x} = 0 \Rightarrow R_{ay} - H_a(-0.06x + 0.6) = 0$$

$$x = 1.8748 \text{ m}$$

$$M = -14.06 \text{ kN.m.}$$

For the maximum positive bending moment in this region occurs at  $D$ ,



$$M_D = R_{ay}5 - H_a(-0.03 \times 25 + 0.6 \times 5) \\ = +25.0 \text{ kN.m}$$

### Span CB

Bending moment at any cross section, in this span is calculated by,

$$M = R_{ay}x - H_a(-0.03x^2 + 0.6x) - 50(x-5) - 10(x-10)\frac{(x-10)}{2}$$

For locating the position of maximum bending moment,

$$\frac{\partial M}{\partial x} = 0 = R_{ay} - H_a(-0.06x + 0.6) - 50 - \frac{10}{2} \times 2(x-10) = 0$$

$$x = 17.5 \text{ m}$$

$$M = 65 \times 17.5 - 133.33(-0.03(17.5)^2 + 0.6(17.5)) - 50(12.5) - \frac{10}{2}(7.5)^2$$

$$M = 56.25 \text{ kN.m}$$

Hence, the maximum positive bending moment occurs in span CB.

### Summary

In this lesson, the arch definition is given. The advantages of arch construction are given in the introduction. Arches are classified as three-hinged, two-hinged and hingeless arches. The analysis of three-hinged arch is considered here. Numerical examples are solved in detail to show the general procedure of three-hinged arch analysis.